Gauging CSO groups in $N = 4$ supergravity

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ABSTRACT: We investigate a class of CSO-gaugings of $\mathcal{N} = 4$ supergravity coupled to 6 vector multiplets. Using the CSO-gaugings we do not find a vacuum that is stable against all scalar perturbations at the point where the matter fields are turned off. However, at this point we do find a stable cosmological scaling solution.

KEYWORDS: Cosmology of Theories beyond the SM, Supergravity Models, Extended Supersymmetry.

Contents

1. Introduction

Gauged supergravity theories have solutions that may provide a stringy explanation of cosmological problems. An interesting example is the fact that gauged $\mathcal{N} = 2$ supergravity has stable De Sitter solutions [1–4]. A second class of cosmologically interesting solutions are the so-called scaling solutions, which might play a role in explaining the accelerating expansion of the universe (see [5, 6] and references therein).

In recent work we have investigated the properties of gauged $\mathcal{N} = 4$ supergravity in four dimensions with the aim of constructing gaugings which lead to a scalar potential which allows positive extremum with nonnegative mass matrix [7, 8]. No such extrema were found. In the present paper we extend this work to contracted groups of the CSO type, and extend the search to include cosmological scaling solutions.

In [8] we limited ourselves to semisimple gauge groups G with dim $G \leq 12$. Then the field content of the four-dimensional theory corresponds to that of the $\mathcal{N} = 1$, $d = 10$

supergravity, which puts the analysis within a string theory context. In [9], we performed a group manifold reduction of the dual version of $\mathcal{N} = 1$, $d = 10$ supergravity, and compared the result with four-dimensional gauged supergravity. For the group manifold SO(3)×SO(3) the resulting gauge group is $CSO(3, 0, 1) \times CSO(3, 0, 1)$. We showed that the effect of this reduction, including nonzero 3-form fluxes, can also be obtained by directly gauging the four-dimensional $\mathcal{N} = 4$ theory with the corresponding CSO group.

In this paper we address CSO-gaugings of $\mathcal{N} = 4$ supergravity with dim $G \leq 12$. $CSO(p, q, r)$ is a contraction of a special orthogonal group: for $r = 0$ they reduce to SO(p, q), if $r \neq 0$ there is an abelian subalgebra of dimension $r(r-1)/2$. To find consistent CSO-gaugings we need to prove a lemma on invariant metrics on CSO-algebras. The main conclusion of the lemma is that only the $CSO(p, q, r)$-groups with $p + q + r = 4$ (we take $r > 0$ in order to have a truly contracted group) give viable gaugings.

We do not present the most general CSO-gauging of $\mathcal{N} = 4$ supergravity. As already mentioned we restrict to 6 vector multiplets. In reference [10] the most general gaugings of $\mathcal{N} = 4$ supergravity are discussed and characterized by a set of parameters $\{\xi_{\alpha M}, f_{\alpha KLM}; \alpha = 1, 2; 1 \leq K, L, M \leq 12\}$ that need to satisfy a set of constraints. Our gaugings correspond to the subset of gaugings where $\xi_{\alpha M} = 0$.

The paper is organized as follows. In section 2 we discuss the scalar fields of $\mathcal{N} = 4$ matter-coupled supergravity. In section 3 we discuss the gauging of $\mathcal{N} = 4$ supergravity coupled to 6 vector multiplets; we briefly review the concept of CSO-groups, or actually their Lie algebras $\mathfrak{cso}(p, q, r)$, present the lemma on invariant metrics on $\mathfrak{cso}$-algebras and discuss the SU(1, 1)-angles. In section 4 we review some results from [8] and in section 5 we apply this to the CSO-gaugings. As in the case of semisimple groups we do not obtain a positive extremum with nonnegative mass matrix. In section 6 we show that a cosmological scaling solution exists in $\mathcal{N} = 4$ CSO gauged supergravity.

2. The scalars of $\mathcal{N} = 4$ supergravity

We consider gauged $\mathcal{N} = 4$ supergravity coupled to n vector multiplets [11]. The scalars parameterize an SO(6, n)/ SO(6) × SO(n) × SU(1, 1)/ U(1) coset and can be split in the $6n$ scalars of the matter multiplets, which parameterize SO(6, n)/ SO(6) × SO(n)-coset, and the two scalars of the supergravity multiplet, which parameterize an SU(1, 1)/ U(1)-coset.

The SU(1, 1)-scalars from the supergravity multiplet are denoted ϕ_α, $\alpha = 1, 2$, and take complex values. When we define $\phi^1 = (\phi_1)^*$ and $\phi^2 = -(\phi_2)^*$, the constraint that restrict them to the SU(1, 1)/ U(1)-coset reads

$$\phi^\alpha \phi_\alpha = |\phi_1|^2 - |\phi_2|^2 = 1. \tag{2.1}$$

A convenient parametrization of the SU(1, 1)-scalars is obtained by using the U(1)-symmetry to take ϕ_1 real:

$$\phi_1 = \frac{1}{\sqrt{1 - r^2}}, \quad \phi_2 = \frac{r e^{i\varphi}}{\sqrt{1 - r^2}}. \tag{2.2}$$

The kinetic term of the SU(1, 1)-scalars then becomes

$$\mathcal{L}_{kin}(r, \varphi) = -\frac{1}{(1-r^2)^2} \left(\partial_\mu r \partial^\mu r + r^2 \partial_\mu \varphi \partial^\mu \varphi \right) . \tag{2.3}$$

The SO(6, n)-scalars from the matter multiplets are denoted $Z_a{}^R$, $a = 1, \ldots, 6$ and $R = 1, \ldots, 6 + n$, and they take real values. The constraint that restricts the SO(6, n)-scalars to the $SO(6, n)/SO(6) \times SO(n)$-coset is

$$Z_a{}^R \eta_{RS} Z_b{}^S = -\delta_{ab} , \tag{2.4}$$

where η_{RS} are the components of the invariant metric in the vector representation of $SO(6, n)$ in a basis such that

$$\eta = \text{diag}(-1, \ldots, -1, +1, \ldots, +1), \tag{2.5}$$

with six negative entries and n positive entries. Hence the scalars $Z_a{}^R$ can be viewed as the upper six rows of $SO(6, n)$-matrix. We define $Z^{RS} = Z_a{}^R Z_a{}^S$ and note that $\delta_R{}^S + 2Z_R{}^S$ is an $SO(6, n)$-matrix, where the indices are raised and lowered with the metric η_{RS}.

In this paper we restrict ourselves to $n = 6$, which makes contact with string theory. From [12] we find a convenient parametrization of the coset $SO(6, 6)/SO(6) \times SO(6)$; we write $Z_a{}^R = (X, Y)_a{}^R$, where X and Y are 6×6-matrices and put

$$\begin{aligned}
X &= \tfrac{1}{2} \left(G + G^{-1} + BG^{-1} - G^{-1}B - BG^{-1}B \right) , \\
Y &= \tfrac{1}{2} \left(G - G^{-1} - BG^{-1} - G^{-1}B - BG^{-1}B \right) ,
\end{aligned} \tag{2.6}$$

where G is an invertible symmetric 6×6-matrix and B is an antisymmetric 6×6-matrix. It is convenient to split the indices $R, S, \ldots$ of η_{RS} in $A, B, \ldots = 1, \ldots, 6$, $(\eta_{AB} = -\delta_{AB})$ and $I, J, \ldots = 7, \ldots, 12$, $(\eta_{IJ} = +\delta_{IJ})$. Hence $Z_a{}^A = X_a{}^A$ and $Z_a{}^I = Y_a{}^{I-6}$. We define a 6×6-matrix containing the independent degrees of freedom of the $SO(6, 6)$-scalars by $P = G + B$ and denote its components by P_{ab}, where $1 \leq a, b, \leq 6$. The kinetic term of the independent scalars P_{ab} then reads:

$$\mathcal{L}_{kin}(P_{ab}) = -\tfrac{1}{2} \partial_\mu P_{ab} \partial^\mu P_{ab} . \tag{2.7}$$

There is a certain freedom in coupling the vector multiplets: for each multiplet, labelled by R, we can introduce an SU(1, 1)-element, of which only a single angle α_R turns out to be important. These angles α_R can be reinterpreted as a modification of the SU(1, 1)-scalars coupling to the multiplet R in the form

$$\phi^1_{(R)} = e^{i\alpha_R} \phi^1 , \quad \phi^2_{(R)} = e^{-i\alpha_R} \phi^2 , \quad \Phi_{(R)} = e^{i\alpha_R} \phi^1 + e^{-i\alpha_R} \phi^2 . \tag{2.8}$$

The kinetic term of the vector fields A_μ^R is

$$\mathcal{L}_{kin}(A_\mu^R) = -\frac{\eta_{RS} + 2Z_{RS}}{4|\Phi_{(R)}|^2} F_{\mu\nu}^R F^{S\mu\nu} , \tag{2.9}$$

where $F_{\mu\nu}^R$ is the nonabelian field strength of A_μ^R.

In this paper we are mainly interested in a special point of the $\mathrm{SO}(6,6)/\mathrm{SO}(6)\times\mathrm{SO}(6)$-manifold, namely the point where the matter multiplets are 'turned off'. This point is denoted Z_0 and corresponds to the identity point of the coset $\mathrm{SO}(6,6)/\mathrm{SO}(6)\times\mathrm{SO}(6)$, that is, $Z_0 \cong \mathrm{SO}(6)\times\mathrm{SO}(6)$. In our parametrization we have at Z_0: $X=\mathbb{1}$, $Y=0$ and $P_{ab}=\delta_{ab}$.

3. CSO gaugings

In the context of maximal supergravities CSO-groups have been used to construct gauged supergravities, see e.g. [13–17]. By truncating the four-dimensional $\mathcal{N}=8$ theory to an $\mathcal{N}=4$ theory one obtains four-dimensional $\mathcal{N}=4$ supergravities with a CSO-gauging [13, 15]. The definition of CSO-algebras as outlined below is similar to the discussion in [16].

Let $\mathfrak{g}$ be a real Lie algebra, then $\mathfrak{g}$ is admissible as a gauge algebra of $\mathcal{N}=4$ supergravity if and only if there exists a basis of generators T_R of $\mathfrak{g}$ such that the structure constants defined by $[T_R,T_S]=f_{RS}{}^U T_U$ satisfy

$$f_{RS}{}^T \eta_{TU} + f_{RU}{}^T \eta_{TS} = 0\,, \tag{3.1}$$

with η as defined in (2.5). We define a symmetric nondegenerate bilinear form Ω on the Lie algebra $\mathfrak{g}$ by its action on the basis elements T_R through

$$\Omega(T_R,T_S) = \eta_{RS}\,. \tag{3.2}$$

The constraint (3.1) then is equivalent to demanding that the form Ω is invariant under the adjoint action of the Lie algebra $\mathfrak{g}$ on itself. From now on we write 'metric' for 'nondegenerate bilinear symmetric form'.

On complex simple Lie algebras there exists only a one-parameter family of invariant metrics and every invariant metric is proportional to the Cartan–Killing metric. For simple real algebras of which the complex extensions is simple there exists up to multiplicative factor only one invariant metric, given by the Cartan–Killing metric. However, for simple real Lie algebras of which the complex extension is not simple, there exists a two-parameter family of invariant metrics. This can be seen from the fact that if the complex extension is not simple, it is of the form $\mathfrak{m}\oplus\mathfrak{m}$, with $\mathfrak{m}$ a complex simple Lie algebra.

For Lie algebras of the type $\mathfrak{cso}(p,q,r)$ (for definitions, see section 3.1) the situation is more delicate. The criterion of nondegeneracy turns out to be very restrictive. We have the following useful lemma:

Lemma on invariant metrics on CSO-algebras. *The Lie algebra $\mathfrak{cso}(p,q,r)$ with $r>0$ admits an invariant nondegenerate symmetric bilinear form (i.e. an invariant metric) only if*

$$(1)\ p+q+r=2 \ \text{ or } \ (2)\ p+q+r=4\,. \tag{3.3}$$

Since the algebras $\mathfrak{cso}(1,0,1) \cong \mathfrak{cso}(0,0,2) \cong \mathfrak{cso}(0,0,4) \cong \mathfrak{u}(1)$ are abelian, the structure constants are zero and give therefore rise to trivial gaugings. Hence, we focus on the CSO-algebras of the type $\mathfrak{cso}(p,q,r)$ with $p+q+r=4$ and $0<r<4$.

3.1 Lie algebras of the type $\mathfrak{cso}(p,q,r)$

In the vector representation the Lie algebra $\mathfrak{so}(p, q + r)$ admits a set of basis elements $J_{AB} = -J_{BA}$, $1 \le A, B \le p + q + r$ satisfying the commutation relation:

$$[J_{AB}, J_{CD}] = g_{BC} J_{AD} + g_{AD} J_{BC} - g_{AC} J_{BD} - g_{BD} J_{AC} \,, \tag{3.4}$$

where g_{AB} are the entries of the diagonal matrix with p eigenvalues $+1$ and $q+r$ eigenvalues -1.

We split the indices[1] $A, B, \ldots$ into indices $I, J, \ldots$ running from 1 to $p + q$ and indices $a, b, \ldots$ running from $p + q + 1$ to $p + q + r$. The Lie algebra $\mathfrak{so}(p, q + r)$ splits as a vector space direct sum $\mathfrak{so}(p, q + r) = \mathfrak{so}(p, q) \oplus \mathcal{V} \oplus \mathcal{Z}$, where the elements J_{IJ} span the $\mathfrak{so}(p, q)$ subalgebra, the elements $J_{Ia} = -J_{aI}$ span the subspace $\mathcal{V}$ and the elements J_{ab} span the subalgebra $\mathcal{Z}$. The subspace $\mathcal{V}$ consists of r copies of the vector representation of the subalgebra $\mathfrak{so}(p, q)$, whereas the subalgebra $\mathcal{Z}$ consists of singlet representations of $\mathfrak{so}(p, q)$. The commutation relations are schematically given by:

$$\begin{aligned}
[\mathfrak{so}(p,q), \mathcal{V}] &\subset \mathcal{V}\,, & [\mathcal{V}, \mathcal{V}] &\subset \mathcal{Z} \oplus \mathfrak{so}(p,q)\,, \\
[\mathfrak{so}(p,q), \mathcal{Z}] &\subset 0\,, & [\mathcal{Z}, \mathcal{V}] &\subset \mathcal{V}\,, \\
[\mathfrak{so}(p,q), \mathfrak{so}(p,q)] &\subset \mathfrak{so}(p,q)\,, & [\mathcal{Z}, \mathcal{Z}] &\subset \mathcal{Z}\,.
\end{aligned} \tag{3.5}$$

We define for any $\xi \in \mathbb{R}$ a linear map $T_\xi : \mathfrak{so}(p, q + r) \to \mathfrak{so}(p, q + r)$ by its action on the subspaces:

$$\begin{aligned}
x &\in \mathfrak{so}(p,q)\,, & T_\xi &: x \mapsto x\,, \\
x &\in \mathcal{V}\,, & T_\xi &: x \mapsto \xi x\,, \\
x &\in \mathcal{Z}\,, & T_\xi &: x \mapsto \xi^2 x\,.
\end{aligned} \tag{3.6}$$

If $\xi \ne 0, \infty$ the map T_ξ is a bijection. The maps T_0 and T_∞ give rise to so-called contracted Lie algebras.

We define the limits $T_0(\mathfrak{so}(p, q)) = \mathfrak{s} \cong \mathfrak{so}(p, q)$, $T_0(\mathcal{V}) = \mathfrak{r}$ and $T_0(\mathcal{Z}) = \mathfrak{z}$. The Lie algebra $\mathfrak{cso}(p, q, r)$ is defined as $T_0(\mathfrak{so}(p, q+r))$. Hence we have $\mathfrak{cso}(p, q, r) = \mathfrak{so}(p, q) \oplus \mathfrak{r} \oplus \mathfrak{z}$ and the commutation rules are of the form

$$[\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{s}\,, \quad [\mathfrak{r}, \mathfrak{r}] \subset \mathfrak{z}\,, \quad [\mathfrak{s}, \mathfrak{r}] \subset \mathfrak{r}\,, \quad [\mathfrak{r}, \mathfrak{z}] = [\mathfrak{s}, \mathfrak{z}] = [\mathfrak{z}, \mathfrak{z}] = 0\,. \tag{3.7}$$

We mention some special cases and properties. If $r = 0$ the construction is trivial and therefore we take $r > 0$. If $p + q = 1$ we have $\mathfrak{s} = 0$ and if $p + q = r = 1$ also $\mathfrak{z} = 0$ and we have $\mathfrak{cso}(1, 0, 1) \cong \mathfrak{cso}(0, 1, 1) \cong \mathfrak{u}(1)$. If $p + q = 2$ the Lie algebra $\mathfrak{s}$ is abelian and if $p + q > 2$ the Lie algebra $\mathfrak{s}$ is semisimple and the vector representation is irreducible. Hence if $p + q > 2$ we have $[\mathfrak{s}, \mathfrak{r}] \cong \mathfrak{r}$. If $r = 1$ we have $\mathfrak{z} = 0$ and the Lie algebra is an Inönü–Wigner contraction.

From the construction follows a convenient set of basis elements of $\mathfrak{cso}(p, q, r)$. The elements $S_{IJ} = -S_{JI}$ are the basis elements of the subalgebra $\mathfrak{s}$, the elements v_{Ia} are the

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[1]The splitting of indices in this case is not related to the splitting of the indices of the SO(6, 6)-scalars introduced in section 2.

basis elements of $\mathfrak{r}$ and the elements $z_{ab} = -z_{ba}$ are the basis elements of $\mathfrak{z}$. The only nonzero commutation relations are:

$$
\begin{aligned}
[S_{IJ}, S_{KL}] &= \tilde{g}_{JK}S_{IL} - \tilde{g}_{IK}S_{JL} - \tilde{g}_{JL}S_{IK} + \tilde{g}_{IL}S_{JK}\,, \\
[S_{IJ}, v_{Ka}] &= \tilde{g}_{JK}v_{Ia} - \tilde{g}_{IK}v_{Ja}\,, \\
[v_{Ia}, v_{Jb}] &= \tilde{g}_{IJ}Z_{ab}\,.
\end{aligned}
\tag{3.8}
$$

The numbers $\tilde{g}_{IJ}$ are the elements of the diagonal metric with p eigenvalues $+1$ and q eigenvalues -1. The commutation relations (3.8) can also be taken as the definition of the Lie algebra $\mathfrak{cso}(p, q, r)$.

3.2 Choosing the $SU(1,1)$-angles

In general the gauge algebra $\mathfrak{g}$ can be decomposed as a direct sum $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \dots$ and it is clear that the $SU(1,1)$-angles can be different on different factors $\mathfrak{g}_i$. With each generator T_R of the gauge algebra $\mathfrak{g}$ we associate a gauge field A_μ^R and an $SU(1,1)$-angle α_R. The gauge group rotates the gauge fields associated to the same factor into each other. All the generators that can be obtained by rotating the generator T_R need to have the same $SU(1,1)$-angle α_R for the gauge group to be a symmetry. Hence along the gauge orbit of T_R, denoted by $\Gamma[T_R]$ and defined by

$$
\Gamma[T_R] = \left\{ \mathrm{e}^{\mathrm{ad}A}(T_R) | A \in \mathfrak{g} \right\}\,,
\tag{3.9}
$$

the $SU(1,1)$-angle has to be constant. If $\Gamma[T_R] \cap \Gamma[T_S] \neq 0$ we need $\alpha_R = \alpha_S$. For semisimple groups the gauge orbits are the simple factors and hence with each simple factor we associate a single $SU(1,1)$-element.

For the algebras $\mathfrak{cso}(2, 0, 2)$, $\mathfrak{cso}(1, 1, 2)$ the gauge orbit of $\mathfrak{s}$, which is one-dimensional, is $\mathfrak{s} \oplus \mathfrak{r}$ and the gauge-orbit of $\mathfrak{r}$ is $\mathfrak{r} \oplus \mathfrak{z}$. For the algebras $\mathfrak{cso}(3, 0, 1)$ and $\mathfrak{cso}(2, 1, 1)$ the gauge orbit of every element of $\mathfrak{s}$ is the whole Lie algebra. Finally, for $\mathfrak{cso}(1, 0, 3)$ the gauge orbit of each element $\mathfrak{r}$ is contained in $\mathfrak{r} \oplus \mathfrak{z}$ and all gauge orbits overlap. Hence for all CSO-type algebras under consideration the $SU(1,1)$-angles have to be constant over the whole Lie algebra $\mathfrak{cso}(p, q, r)$.

3.3 The embedding of CSO-algebras in $SO(6,6)$

The CSO-algebras that are admissible are the $\mathfrak{cso}(p, q, r)$ with $p + q + r = 1$, and since for $r = 0$ the algebra is semisimple, we only consider $r > 0$.

To find a basis such that (3.1) is satisfied on the structure constants we first construct any basis for the Lie algebra and find the invariant metric Ω, which in most cases can be cast in a simple form. The second step is to find a basis-transformation such that in the new basis Ω is diagonalized with all eigenvalues ± 1. Then the structure constants are calculated in this basis, and by construction they satisfy (3.1). This procedure is not unique and it is easy to see that any $SO(6,6)$-transformation on the structure constants leaves the constraint (3.1) invariant. However, we are not trying to be completely exhaustive. On the other hand, an $SO(6,6)$-transformation can be seen as a rotation on the scalar fields

$Z_a{}^R$ and has the physical interpretation of turning on the matter fields (if the rotation is not contained in the subgroup $\mathrm{SO}(6) \times \mathrm{SO}(6)$).

For all CSO-algebras under consideration the dimension is six and the invariant metric has signature $+++---$ (see section 3.4). This implies that precisely two CSO-algebras can be embedded into the vector representation of $\mathrm{SO}(6,6)$.

There is a $\mathbb{Z}_2$-freedom in choosing the embedding into $\mathrm{SO}(6,6)$: for a given invariant metric Ω the eigenvectors with positive eigenvalues can be embedded either in the subspace spanned by the generators T_A where $\eta_{AB} = -\delta_{AB}$ or in the subspace spanned by the generators T_I where $\eta_{IJ} = +\delta_{IJ}$. This difference in embedding can result in a physical difference that modifies the potential. To distinguish between the two kinds of structure constants resulting from the difference in embedding we denote one embedding as $CSO(p,q,r)_+$ and the other as $CSO(p,q,r)_-$. In contrast to the case of semisimple gaugings (where one can use the Cartan–Killing metric to choose a sign-convention), the procedure of assigning a plus or minus to the gauging is arbitrary, since if Ω is an invariant metric, then also $-\Omega$ is an invariant metric that interchanges the plus- and minus-type of gauging. In appendix B we present the structure constants for the different embeddings. We note that for the Lie algebras $\mathfrak{cso}(2,0,2)$ and $\mathfrak{cso}(1,1,2)$ the structure constants of the plus-embedding and minus-embedding are the same, hence no distinction will be made for these algebras.

Having obtained a set of structure constants that satisfies the constraint (3.1) we return to the $\mathcal{N} = 4$ supergravity and use the structure constants as input to investigate the potential. In section 4 we present the details of the potential that are used in the analysis and in section 5 we present the analysis of the potential with the CSO-gaugings. To finish this section, we give a proof of the lemma on invariant metric on CSO-algebras.

3.4 Proof of the lemma

The proof consists of two parts. In the first part (**Part I** below) we prove for all but the CSO-algebras listed in (3.3) that no invariant metric exists. We do this by assuming a bilinear form Ω is invariant and then prove it is degenerate. In the second part (**Part II**) we give the invariant metrics for the CSO-algebras listed in (3.3).

The first part uses the concepts of isotropic subspaces and Witt-indices. For a bilinear symmetric form B on a real vector space V, an isotropic subspace is a subspace W of V on which B vanishes. The maximal isotropic subspace is an isotropic subspace with the maximal dimension. The dimension of the maximal isotropic subspace is the Witt-index of the pair (B, V) and is denoted m_W.

If B is nondegenerate and the dimension of V is n, one can always choose a basis in which B has the matrix form

$$B = \begin{pmatrix} \mathbb{1}_{p \times p} & 0 & \\ 0 & 0 & \mathbb{1}_{r \times r} \\ 0 & \mathbb{1}_{r \times r} & 0 \end{pmatrix}, \quad \text{for } p, r \text{ with } p + 2r = n. \tag{3.10}$$

This clearly shows that the Witt-index is r. Hence we have the inequality: $m_W \leq [n/2]$.

If the center $\mathfrak{z}$ is nonzero we have $[\mathfrak{r}, \mathfrak{r}] = \mathfrak{z}$, that is, for every $z \in \mathfrak{z}$ there are $v_i, w_i \in \mathfrak{r}$ such that $\sum_i [v_i, w_i] = z$. Hence if z, z', with $z = \sum_i [v_i, w_i]$ and $v_i, w_i \in \mathfrak{r}$, we have

$\Omega(z, z') = \sum_i \Omega([v_i, w_i], z') = \sum_i \Omega(v_i, [w_i, z']) = 0$ and hence the center $\mathfrak{z}$ is contained in the maximal isotropic subspace. Hence if the dimension of $\mathfrak{z}$ exceeds half the dimension of the Lie algebra, any invariant symmetric bilinear form is necessarily degenerate.

Part I

We split part I in six different cases. For every case we assume an invariant symmetric bilinear form Ω exists and prove degeneracy. We use the same decomposition as in section 3.1, $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{r} \oplus \mathfrak{z}$, with $\mathfrak{g}$ a CSO-type Lie algebra, and the standard commutation relations (3.8).

$\mathfrak{cso}(p, q, r)$ with $p + q > 2$ and $r > 1$

We have $[\mathfrak{s}, \mathfrak{s}] = \mathfrak{s}$, $[\mathfrak{r}, \mathfrak{r}] = \mathfrak{z}$ and $[\mathfrak{s}, \mathfrak{r}] = \mathfrak{r}$. We prove that $\mathfrak{z}$ is perpendicular to the whole algebra with respect to Ω, which implies that Ω is degenerate.

The center $\mathfrak{z}$ is perpendicular to itself since it is nonzero and thus defines an isotropic subspace. For every $v \in \mathfrak{r}$ there are $j_i \in \mathfrak{s}$ and $w_i \in \mathfrak{r}$ such that $\sum_i [j_i, w_i] = v$. Hence for such v and $z \in \mathfrak{z}$ we have $\Omega(v, z) = \sum_i \Omega(j_i, [w_i, z]) = 0$ and Ω is zero on $\mathfrak{z} \times \mathfrak{r}$. Since $\mathfrak{s}$ is semisimple a similar argument shows that Ω is zero on $\mathfrak{z} \times \mathfrak{s}$ and then $\mathfrak{z}$ is orthogonal to the whole Lie algebra with respect to Ω.

$\mathfrak{cso}(p, q, r)$ with $p + q = 1$ and $r > 3$

We have $\mathfrak{s} = 0$ and $\dim \mathfrak{r} = r$ and $\dim \mathfrak{s} = r(r-1)/2$. The dimension of the center becomes too large for Ω to be nondegenerate if $r(r-1)/2 > r(r+1)/4$. It follows that if $r > 3$ there is no invariant metric.

$\mathfrak{cso}(p, q, r)$ with $p + q = 1$ and $r = 2$

From the commutation relations (3.8) we see that we can choose a basis e, f, z such that the only nonzero commutator is $[e, f] = z$. We have $\Omega(z, z) = 0$, but also $\Omega(e, z) = \Omega(e, [e, f]) = \Omega([e, e], f) = 0$. Similarly $\Omega(z, f) = 0$ and thus z is perpendicular to the whole algebra and Ω is degenerate.

$\mathfrak{cso}(p, q, r)$ with $p + q = 2$ and $r = 1$

The Lie algebras $\mathfrak{cso}(1, 1, 1)$ and $\mathfrak{cso}(2, 0, 1)$ have zero center and hence $[\mathfrak{r}, \mathfrak{r}] = 0$. For every $x \in \mathfrak{r}$ there are $y_i \in \mathfrak{r}$ and $A_i \in \mathfrak{s}$ such that $x = \sum_i [A_i, y_i]$. Therefore we have for such x, y_i, A_i and $v \in \mathfrak{r}$: $\Omega(x, v) = \sum_i \Omega(A_i, [y_i, v]) = 0$. Thus $\mathfrak{r}$ is an isotropic subspace of dimension 2, whereas the dimension of the Lie algebra is 3.

$\mathfrak{cso}(p, q, r)$ with $p + q = 2$ and $r > 2$

We choose a basis $\{j, e_a, f_a, z_{ab}\}$, where $j \in \mathfrak{s}$, $e_a, f_a \in \mathfrak{r}$ and $z_{ab} = -z_{ba} \in \mathfrak{z}$ and $1 \leq a, b \leq r$. In terms of the basis elements in (3.8) we have $j = J_{12}$, $e_a = v_{1a}$, $f = v_{2a}$. The only nonzero commutation relations are

$$[j, e_a] = f_a, \quad [j, f_a] = \sigma e_a, \quad [f_a, f_b] = \sigma z_{ab} \quad [e_a, e_b] = z_{ab}, \tag{3.11}$$

where $\sigma = +1$ for $\mathfrak{cso}(1,1,r)$ and $\sigma = -1$ for $\mathfrak{cso}(2,0,r)$.

From the commutation relations (3.11) one deduces that the subspace spanned by the elements e_a and z_{ab} defines an isotropic subspace of dimension $r(r+1)/2$. The dimension of this isotropic subspace exceeds half the dimension of the Lie algebra if $r > 2$.

$\mathfrak{cso}(p,q,r)$ with $p+q > 3$ and $r = 1$

The Lie algebras in this class have zero center and hence $[\mathfrak{r},\mathfrak{r}] = 0$. We have $\mathfrak{r} = [\mathfrak{s},\mathfrak{r}]$, $\mathfrak{s} = [\mathfrak{s},\mathfrak{s}]$ and $\mathfrak{s}$ is semisimple. It follows that Ω is zero on $\mathfrak{r} \times \mathfrak{r}$ and Ω coincides with the Cartan–Killing metric of $\mathfrak{s}$ on $\mathfrak{s} \times \mathfrak{s}$. Hence we are interested in Ω on $\mathfrak{r} \times \mathfrak{s}$.

From (3.8) we see that we can choose a basis $\{S_{IJ}, v_I\}$, where $1 \leq I, J \leq p+q$, and the only nonzero commutation relations are:

$$
\begin{aligned}
[S_{IJ}, S_{KL}] &= \eta_{JK} S_{IL} - \eta_{IK} S_{JL} - \eta_{JL} S_{IK} + \eta_{IL} S_{JK} \\
[S_{IJ}, v_K] &= \eta_{JK} v_I - \delta_{IK} v_J \,.
\end{aligned}
\tag{3.12}
$$

We define $\Omega_{IJK} = \Omega(v_I, S_{JK}) = -\Omega_{IKJ}$. Invariance requires $\Omega([S_{IJ}, v_K], S_{LM}) = -\Omega(v_K, [S_{IJ}, S_{LM}])$, from which we obtain:

$$
\eta_{JK}\Omega_{ILM} - \eta_{IK}\Omega_{KLM} = -\eta_{JL}\Omega_{KIM} - \eta_{IM}\Omega_{KJL} + \eta_{IL}\Omega_{KJM} + \eta_{JM}\Omega_{KIL} \,.
\tag{3.13}
$$

Contracting equation (3.13) with $\eta^{IK}\eta^{JL}$ we obtain:

$$
\eta^{IJ}\Omega_{IJK} = 0 \,, \forall K \,.
\tag{3.14}
$$

Contracting (3.13) with η^{IK} and using (3.14) we find:

$$
-(p+q)\Omega_{IJK} = \Omega_{IJK} + \Omega_{KIJ} + \Omega_{JIK} \,.
\tag{3.15}
$$

Writing out (3.15) three times with the indices cyclically permuted and adding the three expressions we find the result:

$$
(p+q-3)\,(\Omega_{IJK} + \Omega_{JKI} + \Omega_{KIJ}) = 0 \,.
\tag{3.16}
$$

Since we assumed $p+q > 3$ the cyclic sum of Ω_{IJK} has to vanish.

Using the relation $[S_{IJ}, v^J] = v_I$, where no sum is taken over the repeated index J and where $v^J = \eta^{JK} v_K$, and requiring $\Omega([S_{IJ}, v^J], S_{KL}) = -\Omega(v^J, [S_{IJ}, S_{KL}])$ we obtain:

$$
\Omega_{IJK} + \Omega_{JIK} + \Omega_{KJI} = 0 \,.
\tag{3.17}
$$

Combining (3.17) and the vanishing of the cyclic sum we see that $\Omega_{IJK} = 0$. Hence the subspace $\mathfrak{r}$ is orthogonal to the whole Lie algebra with respect to Ω and Ω is degenerate. This concludes part I.

Part II

We now give for the Lie algebras listed in the lemma on invariant metrics on CSO-algebras the most general invariant metric up to a multiplicative constant.

The Lie algebra $\mathfrak{cso}(1,0,1)$ is abelian and hence any metric is invariant.

For the Lie algebras $\mathfrak{cso}(2,0,2)$ and $\mathfrak{cso}(1,1,2)$ we use the ordered basis $\beta = \{j, e_1, e_2, f_1, f_2, z\}$ with the only nonzero commutation relations

$$[j, e_a] = f_a\,, \quad [j, f_a] = \sigma e_a\,, \quad [f_a, f_b] = \sigma z\,, \quad [e_a, e_b] = z\,, \tag{3.18}$$

where $\sigma = +1$ for $\mathfrak{cso}(1,1,2)$ and $\sigma = -1$ for $\mathfrak{cso}(2,0,2)$.

In the basis β the invariant metric can be written in matrix form as:

$$\Omega = \begin{pmatrix} a & 0 & 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\,, \quad a \in \mathbb{R}\,, \tag{3.19}$$

for both $\mathfrak{cso}(1,1,2)$ and $\mathfrak{cso}(2,0,2)$. The eigenvalues are $-1, -1, +1, +1, \frac{1}{2}(a + \sqrt{a^2 + 4})$, $\frac{1}{2}(a - \sqrt{a^2 + 4})$ and the signature is $+ + + - - -$.

For the Lie algebras $\mathfrak{cso}(2,1,1)$ and $\mathfrak{cso}(3,0,1)$ we use the ordered basis $\beta = \{t_1, t_2, t_3, v_1, v_2, v_3\}$ such that the commutation relations are

$$[t_i, t_j] = \epsilon_{ijk}\eta^{kl}t_l\,, \quad [t_i, v_j] = \epsilon_{ijk}\eta^{kl}v_l\,, \quad [v_i, v_j] = 0\,, \tag{3.20}$$

where ϵ_{ijk} is the three-dimensional alternating symbol and η^{ij} is the diagonal metric with eigenvalues $(+1, -1, -1)$ for $\mathfrak{cso}(2,1,1)$ and with eigenvalues $(+1, +1, +1)$ for $\mathfrak{cso}(3,0,1)$.

With respect to the ordered basis β the invariant metric is given by

$$\Omega = \begin{pmatrix} a\eta & \eta \\ \eta & 0 \end{pmatrix}\,, \tag{3.21}$$

where each entry is a 3×3-matrix. The eigenvalues are $\lambda_\pm = \frac{1}{2}(a \pm \sqrt{a^2 + 4})$, both with multiplicity three, and the signature is $- - - + + +$.

For the Lie algebra $\mathfrak{cso}(1,0,3)$ we use the ordered basis $\beta = \{v_1, v_2, v_3, z_1, z_2, z_3\}$ such that the commutation relations are

$$[v_i, v_j] = \tfrac{1}{2}\epsilon_{ijk}z_k\,, \quad [v_i, z_j] = [z_i, z_j] = 0\,, \tag{3.22}$$

where a summation is understood for every repeated index. The invariant metric is given in matrix form with respect to the basis β by:

$$\Omega = \begin{pmatrix} A_{3\times3} & \mathbb{1}_{3\times3} \\ \mathbb{1}_{3\times3} & 0 \end{pmatrix}\,, \tag{3.23}$$

where $A_{3\times3}$ is an undetermined 3×3-matrix. Since $\det \Omega = -1$ there are no null vectors. We find that if μ_1, μ_2, μ_3 are the eigenvalues of A, then $\lambda_i = \frac{1}{2}\left(\mu_i \pm \sqrt{\mu_i^2 + 4}\right)$ are the eigenvalues of Ω. Hence the signature is $+++---$.

4. The potential and its derivatives

In reference [8] we presented a scheme for analyzing the potential of $\mathcal{N} = 4$ supergravity for semisimple gaugings. We wish to apply this scheme for the CSO-gaugings, since after the analysis of the preceding section the only difference lies in the numerical values of the structure constants. In this section we review the definitions and steps of the analysis of the potential.

4.1 The potential

The analysis of an extremum of the potential can be split in first finding an extremum with respect to the SU(1, 1)-scalars and subsequently investigating whether the point Z_0 determines an extremum with respect to the SO(6, 6)-scalars. We therefore write the potential as:
$$V = \sum_{i,j} \left(R^{(ij)}(\phi)\, V_{ij}(Z) + I^{(ij)}(\phi)\, W_{ij}(Z)\right). \tag{4.1}$$

The indices $i, j, \ldots$ label the different factors in the gauge group G. $R^{(ij)}$ and $I^{(ij)}$ contain the SU(1, 1)-scalars and depend on the gauge coupling constants and the SU(1, 1)-angles, V_{ij} and W_{ij} contain the structure constants, depend on the matter fields, and are symmetric resp. antisymmetric in the indices i, j. The SU(1, 1)-angle associated with the ith factor is written α_i, and the structure constants determined by the ith factor are denoted $f^{(i)\,T}_{RS}$ and we define $f^{(i)}_{RST} = f^{(i)\,U}_{RS}\eta_{TU}$. The functions V_{ij} and W_{ij} are given by:

$$V_{ij} = \tfrac{1}{4}Z^{RU}Z^{SV}(\eta^{TW} + \tfrac{2}{3}Z^{TW})\, f^{(i)}_{RST} f^{(j)}_{UVW}, \tag{4.2}$$
$$W_{ij} = \tfrac{1}{36}\epsilon^{abcdef} Z_a{}^R Z_b{}^S Z_c{}^T Z_d{}^U Z_e{}^V Z_f{}^W\, f^{(i)}_{RST} f^{(j)}_{UVW}. \tag{4.3}$$

The extremum of the potential in the SU(1, 1)-directions has been determined in [7]. For completeness we briefly review this analysis in appendix A. The value of the potential at the extremum with respect to the SU(1, 1)-scalars is given by

$$V_0 = \text{sgn}\, C_- \sqrt{\Delta} - T_-, \tag{4.4}$$

where (see [7])

$$C_- = \sum_{ij} g_i g_j \cos(\alpha_i - \alpha_j) V_{ij}, \tag{4.5}$$

$$T_- = \sum_{ij} a_{ij} W_{ij}, \tag{4.6}$$

$$\Delta = 2 \sum_{ij} \sum_{kl} V_{ij} V_{kl} a_{ik} a_{jl}, \tag{4.7}$$

$$a_{ij} \equiv g_i g_j \sin(\alpha_i - \alpha_j). \tag{4.8}$$

The condition for this extremum to exist is that $\Delta > 0$, which implies that at least two of the $SU(1,1)$-angles must be different.

At the point Z_0 the functions defined above are given by

$$V_{ij}(Z_0) = \delta_{ij}\left(-\tfrac{1}{12}f^{(i)}_{ABC}f^{(i)}_{ABC} + \tfrac{1}{4}f^{(i)}_{ABI}f^{(i)}_{ABI}\right), \tag{4.9}$$

$$W_{ij}(Z_0) = \tfrac{1}{36}\epsilon^{ABCDEF}f^{(i)}_{ABC}f^{(j)}_{DEF}, \tag{4.10}$$

$$\Delta(Z_0) = 2\sum_{i,j}a_{ij}^2 V_{ii}(Z_0)V_{jj}(Z_0), \tag{4.11}$$

$$C_-(Z_0) = \sum_i g_i^2 V_{ii}(Z_0), \tag{4.12}$$

$$R^{(ij)}(Z_0) = \delta_{ij}\frac{2\,\mathrm{sign}\,C_-(Z_0)}{\sqrt{\Delta_0}}\sum_j V_{jj}(Z_0)a_{ij}^2, \tag{4.13}$$

$$I^{(ij)}(Z_0) = -a_{ij}. \tag{4.14}$$

With the formulae (4.9- 4.14) it is easy to plug in the values of the structure constants and determine the value of the potential at Z_0, see section 5.

4.2 The derivatives of the potential

To determine whether the point Z_0 is an extremum with respect to the $SO(6,6)$-scalars we calculate the derivatives with respect to the parameters P_{ab} introduced in section 2 (see also [8]). We have

$$\frac{\partial V}{\partial P_{ab}}(Z_0) = \sum_i R^{(ii)}(Z_0)f^{(i)}_{a+6,CJ}f^{(i)}_{bCJ} - \tfrac{1}{6}\sum_{ij}a_{ij}\epsilon^{bCDEFG}f^{(i)}_{a+6,CD}f^{(j)}_{EFG}. \tag{4.15}$$

Since for CSO-gaugings at most two groups are possible to fit in $SO(6,6)$ the summations over the indices i,j simplify significantly. For the point Z_0 to be an extremum the 6×6-matrix $\partial V/\partial P$ should vanish.

If the point Z_0 turns out to be an extremum with respect to both the $SU(1,1)$-scalars and the $SO(6,6)$-scalars, we need the second derivatives at Z_0 to determine whether the extremum is stable or unstable. Schematically the second derivatives are given by

$$\frac{\partial^2 V}{\partial\phi^2} = \sum_{ij}\frac{\partial^2 R^{(ij)}}{\partial\phi^2}V_{ij}, \tag{4.16}$$

$$\frac{\partial^2 V}{\partial\phi\partial P} = \sum_{ij}\frac{\partial R^{(ij)}}{\partial\phi}\frac{\partial V_{ij}}{\partial P}, \tag{4.17}$$

$$\frac{\partial^2 V}{\partial P^2} = \sum_{ij}R^{(ij)}\frac{\partial^2 V_{ij}}{\partial P^2} + I^{(ij)}\frac{\partial^2 W_{ij}}{\partial P^2}. \tag{4.18}$$

The second derivatives (4.16) were studied in [8]. The sign of (4.16) depends on the sign of C_-. For positive (negative) C_- the extremum in the $SU(1,1)$-scalars is a minimum (maximum). The mixed second derivatives vanish if either the derivatives with respect to the $SU(1,1)$-scalars ϕ or with respect to the matter scalars vanishes.

gauge factor	$V_{ii}(Z_0)$	gauge factor	$V_{ii}(Z_0)$
$CSO(1,0,3)_+$	1	$CSO(1,0,3)_-$	1
$CSO(2,0,2)_+$	0	$CSO(2,0,2)_-$	0
$CSO(1,1,2)_+$	1	$CSO(1,1,2)_-$	1
$CSO(3,0,1)_+$	$-\frac{1}{2}(\lambda^4 + 4\lambda^2 + 1)$	$CSO(3,0,1)_-$	$-\frac{1}{2}(\lambda^4 + 4\lambda^2 + 1)$
$CSO(2,1,1)_+$	$\frac{1}{2}(3\lambda^4 + 4\lambda^2 + 3)$	$CSO(2,1,1)_-$	$\frac{1}{2}(3\lambda^4 + 4\lambda^2 + 3)$

Table 1: The value of V_{ii} at the point Z_0 for different gauge factors. The plus- and minus-sign refer to two distinct possibilities to embed the factor into the gauge group. The number λ is an arbitrary positive number, coming from an arbitrary constant in the invariant metric.

Hence if $C_- > 0$ at Z_0 we need to check the eigenvalues of the matrix of second derivatives (4.18). With the formulas of reference [8] it is a matter of algebra to obtain:

$$
\begin{aligned}
\frac{\partial^2 V_{ij}}{\partial P_{ab}\partial P_{cd}}(Z_0) =& 0\,, \quad i \neq j\,, \\
\frac{\partial^2 V_{ii}}{\partial P_{ab}\partial P_{cd}}(Z_0) =& \delta_{ac} f^{(i)}_{bGJ} f^{(i)}_{dGJ} + \delta_{bd} f^{(i)}_{a+6,GJ} f^{(i)}_{cGJ} - \tfrac{1}{2}\delta_{bc} f^{(i)}_{a+6,GJ} f^{(i)}_{dGJ} \\
& - \tfrac{1}{2}\delta_{ad} f^{(i)}_{c+6,GJ} f^{(i)}_{bGJ} + f^{(i)}_{a+6,c+6,R} f^{(i)}_{b,d,J} + f^{(i)}_{b,c+6,R} f^{(i)}_{a+6,d,R}\,, \\
\frac{\partial^2 W_{ij}}{\partial P_{ab} P_{cd}}(Z_0) =& \tfrac{1}{24}\epsilon^{bBCDEF}\left(\delta_{ac} f^{(i)}_{dBC} f^{(j)}_{DEF} - \delta_{ad} f^{(i)}_{c+6,BC} f^{(j)}_{DEF} \right) \\
& + \tfrac{1}{24}\epsilon^{dBCDEF}\left(\delta_{ac} f^{(i)}_{bBC} f^{(j)}_{DEF} - \delta_{bc} f^{(i)}_{a+6,BC} f^{(j)}_{DEF} \right) \\
& + \tfrac{1}{12}\epsilon^{bdCDEF}\left(2 f^{(i)}_{a+6,c+6,C} f^{(j)}_{DEF} + 3 f^{(i)}_{a+6,CD} f^{(j)}_{c+6,EF} \right) \\
& - (i \leftrightarrow j)\,.
\end{aligned}
\tag{4.19}
$$

The stability is then determined by the eigenvalues of the 36×36-matrix given by

$$
\sum_i R^{(ii)}(Z_0) \frac{\partial^2 V_{ii}}{\partial P_{ab}\partial P_{cd}}(Z_0) - \sum_{ij} a_{ij} \frac{\partial^2 W_{ij}}{\partial P_{ab} P_{cd}}(Z_0)\,.
\tag{4.20}
$$

5. Analysis of the potentials of CSO-gaugings

With the formulas of section 4 and the structure constants of the CSO-algebras, given in appendix B, at our disposal, we analyze the potential and the first and second derivatives at Z_0 for different CSO-gaugings. For each gauge group the function $V_{ii}(Z_0)$ is given in table 1. Note that the value of $V_{ii}(Z_0)$ is the same for plus- and minus-embeddings. Since only two CSO gauge-algebras fit into SO(6,6) we have $\Delta(Z_0) = 4a_{12}^2 V_{11}(Z_0)V_{22}(Z_0)$, hence $\Delta > 0$ if and only if both $V_{11}(Z_0)$ and $V_{22}(Z_0)$ are nonzero and have the same sign. Hence in searching for gaugings that admit an extremum with respect to the SU(1,1)-scalars, we can disregard the gaugings that involve $CSO(2,0,2)$ and the gaugings of which precisely one factor is $CSO(3,0,1)$.

For the groups $CSO(3,0,1)$ and $CSO(2,1,1)$ the structure constants contain an undetermined positive parameter λ that cannot be removed redefinition of the generators

preserving the constraints (3.1). This parameter is a remnant of the invariant metric; there is in general an m-parameter family of invariant metrics with $m > 1$ for $\mathfrak{cso}(p, q, r)$ with $p + q + r = 4$.

The gaugings for which the point Z_0 corresponds to an extremum of both the $SU(1, 1)$- and the $SO(6, 6)$-scalars are: $CSO(1, 0, 3)_- \times CSO(1, 0, 3)_-$, $CSO(1, 0, 3)_+ \times CSO(1, 0, 3)_-$ and $CSO(1, 0, 3)_+ \times CSO(1, 0, 3)_+$. Only these gaugings have vanishing derivative with respect to the parameters P_{ab} and $\Delta > 0$. For these three gaugings the value of the potential at the point Z_0 is given by $V_0 = 0$. With respect to the $SU(1, 1)$-scalars the potential is a minimum, $C_-(Z_0) > 0$, but with respect to the $SO(6, 6)$-scalars the extremum is unstable; the mass-matrix $\partial^2 V / \partial P \partial P$ has both positive and negative eigenvalues.

6. Cosmological scaling solutions

If a scalar potential is of the form

$$V(\chi, \Phi_i) = e^{b\chi}\, U(\Phi_i)\,, \tag{6.1}$$

where χ has canonical kinetic term and is independent of the scalars Φ_i, a cosmological scaling solution exists if the function $U(\Phi_i)$ has a positive extremum with respect to the scalars Φ_i [5]. The scale factor of the Friedmann-Robertson-Walker metric goes as t^{1/b^2} for the scaling solution. The characteristic feature of scaling solutions is that the ratio of the kinetic energy of the scalar χ and the potential energy of the scalar χ remains constant during evolution. Scaling solutions appear as fixed points in autonomous systems that describe scalar cosmologies, see [18] for a recent review and a list of references.

In $\mathcal{N} = 4$ supergravity the potential factorizes in a trivial way if all $SU(1, 1)$-angles are equal; in this case the function $R^{(ij)}(r, \varphi)$ simplifies to:

$$R^{(ij)}(r, \varphi) = g_i g_j \frac{1 + r^2 - 2r\cos\varphi}{1 - r^2} = g_i g_j \frac{|1 + z|^2}{1 - |z|^2}\,, \tag{6.2}$$

where $z = -re^{i\varphi}$. Introducing $\tau = i(1 - z)/(1 + z)$, which takes values in the complex upper half plane since $|z| < 1$, and $\sigma = \mathrm{Re}\,\tau$ and $e^{-\chi} = \mathrm{Im}\,\tau$ one finds

$$R^{(ij)}(\chi, \sigma) = g_i g_j e^{\chi}\,. \tag{6.3}$$

Hence we find for the potential at Z_0 in this case

$$V(Z_0) = -\tfrac{1}{12} e^{\chi} \sum_i g_i^2 \left(f_{ABC}^{(i)} f_{ABC}^{(i)} - 3 f_{ABI}^{(i)} f_{ABI}^{(i)} \right)\,. \tag{6.4}$$

The first derivatives with respect to P_{ab} at Z_0 simplifies to:

$$\frac{\partial V}{\partial P_{ab}}(Z_0) = e^{\chi} \sum_i g_i^2 f_{a+6,DK}^{(i)} f_{bDK}^{(i)}\,. \tag{6.5}$$

The second derivatives with respect to P_{ab} at Z_0 become:

$$\frac{\partial^2 V}{\partial P_{ab} \partial P_{cd}}(Z_0) = e^{\chi} \sum_i g_i^2 \Big(\delta_{ac} f_{bCJ}^{(i)} f_{dCJ}^{(i)} + \delta_{bd} f_{a+6,CJ}^{(i)} f_{c+6,CJ}^{(i)}$$
$$- \tfrac{1}{2} \delta_{bc} f_{a+6,CJ}^{(i)} f_{dCJ}^{(i)} - \tfrac{1}{2} \delta_{ad} f_{c+6,CJ}^{(i)} f_{bCJ}^{(i)} + 2 f_{a+6,c+6,R}^{(i)} f_{bdR}^{(i)} \Big) \tag{6.6}$$

The computations are simplified by noting that the formulas factorize into contributions of different factor groups. Hence to look for an extremum one only has to investigate the contributions of different factor groups to $\partial V / \partial P$.

We find that only $CSO(1,1,2)$ has vanishing contribution to $\partial V / \partial P$ and hence we find that the CSO-gaugings that allow for scaling solutions at Z_0 are $CSO(1,1,2)$ and $CSO(1,1,2) \times CSO(1,1,2)$. Note that the structure constants of $CSO(1,1,2)_+$ are the same as of $CSO(1,1,2)_-$. For the gauging $CSO(1,1,2) \times CSO(1,1,2)$ the eigenvalues of $\partial^2 V / \partial P^2$ are found to be all positive. The potential at Z_0 is given by:

$$V(\chi, Z_0) = (g_1^2 + g_2^2) \, \mathrm{e}^\chi \, . \tag{6.7}$$

Hence the gauging $CSO(1,1,2) \times CSO(1,1,2)$ admits a stable scaling solution. The same is then true for the gauging $CSO(1,1,2)$, since this is a truncation of the gauging $CSO(1,1,2) \times CSO(1,1,2)$ obtained by putting $g_2 = 0$.

7. Conclusions

The conclusions of this paper can be split in three parts.

The first conclusion concerns the gaugings in matter-coupled $\mathcal{N} = 4$ supergravity with CSO-groups. In the formulation of $\mathcal{N} = 4$ supergravity of [11] the only possible CSO-gaugings require that the Lie algebra $\mathfrak{cso}(p,q,r)$ admits an invariant metric. The only Lie algebras $\mathfrak{cso}(p,q,r)$ with $r > 0$ that admit an invariant metric are those with $p + q + r = 2, 4$. If $p + q + r = 2$ the Lie algebra $\mathfrak{cso}(p,q,r)$ is abelian and hence we considered only $p + q + r = 4$.

The second conclusion is that the CSO-gaugings that we considered showed no stable minimum with respect to all $36 + 2$ scalars at the point Z_0. This analysis concerns the case of $\mathcal{N} = 4$ supergravity with six vectormultiplets, and is therefore not completely general. Also the formalism used in the present paper and in [8] has recently been generalized [10]. Going beyond the present paper as proposed in [10] involves solving a system of constraints involving parameters $\{\xi_{\alpha M}, \, f_{\alpha KLM}\}$. It is an interesting and important challenge to solve these equations for $\xi_{\alpha M} \neq 0$, and to perform a general analysis of scalar potentials in gauged $\mathcal{N} = 4$ supergravity.

The third conclusion is that a stable scaling solution exists at Z_0 in $\mathcal{N} = 4$ gauged supergravity with gauge group $CSO(1,1,2)$, or any power of $CSO(1,1,2)$. The scaling solution is characterized by a scale factor, which grows linearly in time and the effective potential contains one scalar χ;

$$V_{eff}(\chi) = (g_1^2 + g_2^2 + \ldots) \, \mathrm{e}^\chi \, . \tag{7.1}$$

The numbers g_i are the coupling constants for each factor of $CSO(1,1,2)$. Also this analysis is not exhaustive. For example, there might be scalars in the $SO(6,6)$-sector that factorize out of the potential such as to combine with the $SU(1,1)$-scalar an overall exponential factor. It will be interesting to study the cosmological models resulting from these scaling solutions in more detail.

$\mathfrak{cso}(\mathbf{2,0,2})$				
$f_{12}{}^3 = 1$	$f_{13}{}^2 = -1$	$f_{18}{}^9 = 1$	$f_{19}{}^8 = -1$	$f_{23}{}^1 = 1$
$f_{23}{}^7 = -1$	$f_{27}{}^3 = -1$	$f_{37}{}^2 = 1$	$f_{78}{}^9 = 1$	$f_{79}{}^8 = -1$
$f_{89}{}^1 = -1$	$f_{56}{}^7 = 1$			
$\mathfrak{cso}(\mathbf{1,1,2})$				
$f_{12}{}^8 = -1$	$f_{13}{}^9 = 1$	$f_{18}{}^2 = -1$	$f_{19}{}^3 = 1$	$f_{27}{}^8 = 1$
$f_{28}{}^1 = 1$	$f_{28}{}^7 = -1$	$f_{37}{}^9 = -1$	$f_{39}{}^1 = -1$	$f_{39}{}^7 = 1$
$f_{78}{}^2 = -1$	$f_{79}{}^3 = 1$			
$\mathfrak{cso}(\mathbf{1,0,3})$				
$f_{12}{}^3 = 1$	$f_{12}{}^9 = -1$	$f_{18}{}^9 = -1$	$f_{13}{}^2 = -1$	$f_{13}{}^8 = 1$
$f_{18}{}^3 = 1$	$f_{19}{}^2 = -1$	$f_{19}{}^8 = 1$	$f_{23}{}^1 = 1$	$f_{23}{}^7 = -1$
$f_{27}{}^3 = -1$	$f_{27}{}^9 = 1$	$f_{29}{}^1 = 1$	$f_{29}{}^7 = -1$	$f_{37}{}^2 = 1$
$f_{27}{}^8 = -1$	$f_{38}{}^1 = -1$	$f_{38}{}^7 = 1$	$f_{78}{}^3 = 1$	$f_{78}{}^9 = -1$
$f_{79}{}^2 = -1$	$f_{79}{}^5 = 1$	$f_{89}{}^1 = 1$	$f_{89}{}^7 = -1$	
$\mathfrak{cso}(\mathbf{2,1,1})$				
$f_{12}{}^3 = -\lambda^2$	$f_{12}{}^9 = -1$	$f_{13}{}^2 = \lambda^2$	$f_{13}{}^8 = 1$	$f_{18}{}^3 = 1$
$f_{18}{}^9 = -(\lambda^2 + 2)$	$f_{19}{}^2 = -1$	$f_{19}{}^8 = (\lambda^2 + 2)$	$f_{23}{}^1 = -\lambda^2$	$f_{23}{}^7 = 2\lambda^2 + 1$
$f_{27}{}^3 = 2\lambda^2 + 1$	$f_{27}{}^9 = -\lambda^2$	$f_{29}{}^1 = 1$	$f_{29}{}^7 = \lambda^2$	$f_{37}{}^2 = -(2\lambda^2 + 1)$
$f_{37}{}^8 = \lambda^2$	$f_{38}{}^1 = -1$	$f_{38}{}^7 = \lambda^2$	$f_{78}{}^3 = -\lambda^2$	$f_{78}{}^9 = -1$
$f_{79}{}^2 = \lambda^2$	$f_{79}{}^5 = 1$	$f_{89}{}^1 = \lambda^2 + 2$	$f_{89}{}^7 = -1$	

Table 2: Structure constants of some relevant $\mathfrak{cso}$-algebras.

Acknowledgments

A. SU(1, 1) scalars and angles

When we parameterize the coset $\mathrm{SU}(1,1)/\mathrm{U}(1)$ as in eq. (2.2), the scalars r and φ appear in the potential (4.1) through

$$R^{(ij)} = \frac{g_i g_j}{2}(\Phi_i^* \Phi_j + \Phi_j^* \Phi_i)$$
$$= g_i g_j \left(\cos(\alpha_i - \alpha_j)\frac{1+r^2}{1-r^2} - \frac{2r}{1-r^2}\cos(\alpha_i + \alpha_j + \varphi) \right), \tag{A.1}$$

$$I^{(ij)} = \frac{g_i g_j}{2i}(\Phi_i^* \Phi_j - \Phi_j^* \Phi_i) = -g_i g_j \sin(\alpha_i - \alpha_j). \tag{A.2}$$

Introducing

$$C_\pm = \sum_{ij} g_i g_j \cos(\alpha_i \pm \alpha_j) V_{ij}\,, \quad S_+ = \sum_{ij} g_i g_j \sin(\alpha_i + \alpha_j) V_{ij}\,, \qquad (A.3)$$

$$T_- = \sum_{ij} g_i g_j \sin(\alpha_i - \alpha_j) W_{ij}\,, \qquad (A.4)$$

we rewrite the potential as

$$V = C_- \frac{1 + r^2}{1 - r^2} - \frac{2r}{1 - r^2} \left(C_+ \cos\varphi - S_+ \sin\varphi \right) - T_-\,. \qquad (A.5)$$

This extremum in r and φ takes on the form

$$\cos\varphi_0 = \frac{s_1 C_+}{\sqrt{C_+^2 + S_+^2}}\,, \quad \sin\varphi_0 = -\frac{s_1 S_+}{\sqrt{C_+^2 + S_+^2}}\,,$$

$$r_0 = \frac{1}{\sqrt{C_+^2 + S_+^2}} \left(s_1 C_- + s_2 \sqrt{\Delta} \right)\,, \qquad \Delta \equiv C_-^2 - C_+^2 - S_+^2\,, \qquad (A.6)$$

where s_1 and s_2 are signs. These are determined by requiring $0 \le r_0 < 1$, this gives $s_1 = \mathrm{sgn}\, C_-$ and $s_2 = -1$. Substitution of r_0 and φ_0 in V leads to eq. (4.4).

In the case that all SU(1,1) angles α_i vanish, $S_+ = T_- = 0$ and $C_- = C_+$, and one finds $r_0 = 1$ and $\Delta = 0$. This is a singular point of the parametrization, which we will exclude. This case corresponds to the Freedman-Schwarz potential [19], which has no minimum.

For the kinetic term and mass-matrix of the SU(1,1)-scalars we introduce:

$$x = \frac{2}{(1 - r_0)^2}(r\cos\varphi - r_0 \cos\varphi_0)\,,$$

$$y = \frac{2}{(1 - r_0)^2}(r\sin\varphi - r_0 \sin\varphi_0)\,. \qquad (A.7)$$

In these variables we find

$$\mathcal{L}(x, y) = -\tfrac{1}{2} \left(\frac{1 - r_0^2}{1 - r^2} \right)^2 \left((\partial x)^2 + (\partial y)^2 \right) - V_0$$

$$-\tfrac{1}{2} \mathrm{sgn}\, C_- \sqrt{\Delta}\, (x^2 + y^2) + \dots\,, \qquad (A.8)$$

where the ellipsis indicate terms of higher order in x and y.

B. Structure constants

In this appendix we give the structure constants of the $\mathfrak{cso}(p, q, r)$ Lie algebras with $p + q + r = 4$ in a basis such that the constraint (3.1) is satisfied. The Lie algebras $\mathfrak{cso}(p, q, r)$ with $p + q + r = 4$ have dimension six and the invariant metric has signature $+++---$. A gauge algebra consists of two Lie algebras $\mathfrak{cso}(p, q, r)$ with $p + q + r = 4$, and the first Lie algebra can be embedded into the subspace spanned by the generators $T_1, T_2, T_3, T_7, T_8, T_9$ and the second can embedded into the subspace spanned by the generators $T_4, T_5, T_6, T_{10}, T_{11}, T_{12}$.

We give the structure constants of every $\mathfrak{cso}(p,q,r)$ with $p + q + r = 4$ as embedded in the subspace spanned by the generators $T_1, T_2, T_3, T_7, T_8, T_9$ since the other embedding can be obtained from the latter by the following permutation of the indices: $\sigma = (14)(25)(36)(7\ 10)(8\ 11)(9\ 12) \in S_{12}$. In fact we also only give the structure constants of the plus-embedding, the minus-embedding (with the generators lying in the same subspace) can be obtained by applying the following permutation of the indices: $\tau = (17)(28)(39)(4\ 10)(5\ 11)(6\ 12) \in S_{12}$. Consistency requires $\sigma\tau = \tau\sigma$, which is easily seen to be satisfied.

With these preliminaries the structure constants of $\mathfrak{cso}(2,0,2)$, $\mathfrak{cso}(1,1,2)$ and $\mathfrak{cso}(1,0,3)$ are given as in table 2. To be economic in writing we only present the nonzero structure constants $f_{RS}{}^T$ for which $R < S$.

The number λ is related to the undetermined constant a in the invariant metric of $\mathfrak{cs}(2,1,1)$ and $\mathfrak{cso}(3,0,1)$ by $2\lambda = a + \sqrt{a^2 + 4}$. Since the function $x \mapsto x + \sqrt{x^2 + 4}$ is one-to-one from $\mathbb{R}$ to the set of positive real numbers, the number λ can be considered an arbitrary positive real number.

The totally antisymmetric tensors $f_{ABC} = f_{AB}{}^D \eta_{DC}$ of $\mathfrak{cso}(3,0,1)$ are more easily displayed in tensor form:

$$f_{ABC} = -(\lambda^2 + 2)\epsilon_{ABC}\,, \quad f_{ABI} = -\epsilon_{AB(I-6)}\,,$$
$$f_{AIJ} = \lambda^2 \epsilon_{A(I-6)(J-6)}\,, \quad f_{IJK} = (2\lambda^2 + 1)\epsilon_{(I-6)(J-6)(K-6)}\,, \tag{B.1}$$

where $\epsilon_{abc} = +1(-1)$ if (abc) is an even (odd) permutation of (123), and otherwise it is zero.

References

[1] P. Fré, M. Trigiante and A. Van Proeyen, *Stable de Sitter vacua from $N = 2$ supergravity*, *Class. and Quant. Grav.* **19** (2002) 4167 [hep-th/0205119].

[2] R. D'Auria and S. Vaulà, *$D = 6$, $N = 2$, $F(4)$-supergravity with supersymmetric de Sitter background*, *JHEP* **09** (2002) 057 [hep-th/0203074].

[3] K. Behrndt and S. Mahapatra, *de Sitter vacua from $N = 2$ gauged supergravity*, *JHEP* **01** (2004) 068 [hep-th/0312063].

[4] B. Cosemans and G. Smet, *Stable de Sitter vacua in $N = 2$, $D = 5$ supergravity*, *Class. and Quant. Grav.* **22** (2005) 2359 [hep-th/0502202].

[5] N.G. Sanchez, *The black hole: scatterer, absorber and emitter of particles*, hep-th/9711068.

[6] A.R. Liddle and R.J. Scherrer, *A classification of scalar field potentials with cosmological scaling solutions*, *Phys. Rev.* **D 59** (1999) 023509 [astro-ph/9809272].

[7] M. de Roo, D.B. Westra and S. Panda, *de Sitter solutions in $N = 4$ matter coupled supergravity*, *JHEP* **02** (2003) 003 [hep-th/0212216].

[8] M. de Roo, D.B. Westra, S. Panda and M. Trigiante, *Potential and mass-matrix in gauged $N = 4$ supergravity*, *JHEP* **11** (2003) 022 [hep-th/0310187].

[9] M. de Roo, M.G.C. Eenink, D.B. Westra and S. Panda, *Group manifold reduction of dual $N = 1$ $D = 10$ supergravity*, *JHEP* **06** (2005) 077 [hep-th/0503059].

[10] J. Schön and M. Weidner, *Gauged N = 4 supergravities*, *JHEP* **05** (2006) 034 [hep-th/0602024].

[11] M. de Roo, *Matter coupling in N = 4 supergravity*, *Nucl. Phys.* **B 255** (1985) 515;
M. de Roo and P. Wagemans, *Gauge matter coupling in N = 4 supergravity*, *Nucl. Phys.* **B 262** (1985) 644.

[12] J. Maharana and J.H. Schwarz, *Noncompact symmetries in string theory*, *Nucl. Phys.* **B 390** (1993) 3 [hep-th/9207016].

[13] C.M. Hull, *More gaugings of N = 8 supergravity*, *Phys. Lett.* **B 148** (1984) 297.

[14] C.M. Hull, *Noncompact gaugings of N = 8 supergravity*, *Phys. Lett.* **B 142** (1984) 39.

[15] C.M. Hull and N.P. Warner, *The structure of the gauged N = 8 supergravity theories*, *Nucl. Phys.* **B 253** (1985) 650.

[16] L. Andrianopoli, F. Cordaro, P. Fré' and L. Gualtieri, *Non-semisimple gaugings of D = 5 N = 8 supergravity and FDAS*, *Class. and Quant. Grav.* **18** (2001) 395 [hep-th/0009048].

[17] L. Andrianopoli, R. D'Auria, S. Ferrara and M.A. Lledo, *Gauging of flat groups in four dimensional supergravity*, *JHEP* **07** (2002) 010 [hep-th/0203206].

[18] E.J. Copeland, M. Sami and S. Tsujikawa, *Dynamics of dark energy*, hep-th/0603057.

[19] D.Z. Freedman and J.H. Schwarz, *N=4 supergravity theory with local* SU(2) × SU(2) *invariance*, *Nucl. Phys.* **B 137** (1978) 333.